Theoretical Analysis of Balloon

Zhang Chang-kai

March 14, 2016

1 Original Theory

This section is for the introduction of some concepts in nonlinear continuum mechanics and thermoelastic theory.

1.1 Deformation Gradient

Suppose the displacement vector of a particle in the undeformed state and the deformed state is $\boldsymbol{x}$ and $\boldsymbol{X}$, and then there exists[1]

$$d\boldsymbol{x} = \frac{\partial \boldsymbol{x}}{\partial \boldsymbol{X}} d\boldsymbol{X} \tag{1}$$

Thus there is a transformation from the undeformed state to the deformed state. So we have the definition

Definition 1. *The* ***deformation gradient*** *of a transforming material is defined as*

$$\boldsymbol{F} = \frac{\partial \boldsymbol{x}}{\partial \boldsymbol{X}} \tag{2}$$

This is like the ratio of length in the two states L/L_0.

1.2 Left Cauchy-Green Deformation Gradient

We need to relate the stress tensor with the deformation tensor in application. However, the deformation gradient might be not symmetric while the stress is symmetric. Thus define a symmetric tensor[1]

$$\boldsymbol{B} = \boldsymbol{F}\boldsymbol{F}^{\boldsymbol{T}} \tag{3}$$

The product in the right hand side is the product of its matrix representation.

1.3 Strain Energy Density Function

Every transformation of material will store a sort of energy, such as the Young's deformation energy density[2]

$$W = \frac{1}{2} E \varepsilon^2 \tag{4}$$

Where E is the Young's modulus and ε is defined as $\varepsilon = \Delta L/L_0$, just as the deformation gradient.

The same as above, there is a general version of energy density[3]

$$W = W(\boldsymbol{F})$$

1.4 Stress-Strain Relation of Hyperelastic Material

Just as what is renown in the Young's energy density and stress, consider the Lagrangian formulated as

$$\mathcal{L} = K(\boldsymbol{F}) - W(\boldsymbol{F}) \tag{5}$$

Where the deformation gradient is regarded as the general coordinates. Thus, considering the steady state which annihilates kinetic energy, we define the 1st Piola-Kirchhoff stress tensor as[4]

$$\boldsymbol{P} = \frac{d\vec{F}}{d\vec{a}} = -\frac{\partial \mathcal{L}}{\partial \boldsymbol{F}} = \frac{\partial W(\boldsymbol{F})}{\partial \boldsymbol{F}} \tag{6}$$

Here, the 1st Piola-Kirchhoff stress tensor aims at the stress exerting on the material *relative to reference configuration* (the original state). And according to[1]

$$dA\,\boldsymbol{n} = J da \boldsymbol{F}^{-T} \cdot \boldsymbol{N} \tag{7}$$

We can generate the Cauchy stress tensor

$$\boldsymbol{\sigma} = \frac{d\vec{F}}{d\vec{A}} = \frac{1}{J}\frac{\partial W}{\partial \boldsymbol{F}} \cdot \boldsymbol{F}^T = \frac{2}{J}\boldsymbol{B} \cdot \frac{\partial W}{\partial \boldsymbol{B}} \tag{8}$$

1.5 Invariants of Tensors

Definition 2. *The invariants of a tensor[5] is defined as the coefficient of the characteristic polynomial of the tensor, i.e., the coefficient of*

$$p(\lambda) = \det(\boldsymbol{A} - \lambda \boldsymbol{E}) \tag{9}$$

Theorem 1. *If a function of tensors is invariant under the rotation, which is called the material is isotropic, then the function can be expressed as the main invariants of the tensor.*

Usually, for a three dimensional tensor, we have

$$I_1 = \mathrm{tr}\boldsymbol{B}, \;\; I_2 = \frac{1}{2}\left[(\mathrm{tr}\boldsymbol{B})^2 - \mathrm{tr}(\boldsymbol{B})^2\right], \;\; J = \det \boldsymbol{B} \tag{10}$$

2 Application to Hyperelastic Model

2.1 Mooney-Rivlin Model

The Mooney-Rivlin Model[6] is the simplest instance of polynomial hyperelastic model, which constructs the strain energy density as a polynomial of the three invariants of left Cauchy-Green deformation gradient I_1, I_2 and J as

$$W = \sum_{i,j=0}^{n} C_{pq}(\bar{I}_1 - 3)^i(\bar{I}_2 - 3)^j + \sum_{k=1}^{m} D_k(J-1)^k \tag{11}$$

When Mooney-Rivlin Model is the situation that there are no other terms except for $i = 0,\ j = 1$; $i = 1,\ j = 0$; and $k = 1$.

However, we will use a simpler model for the rubber is usually incompressible and thus $J = 1$ will hold for every rubber material. Thus, we write the energy density as

$$W = C_1(I_1 - 3) + C_2(I_2 - 3) + D(J - 1) \tag{12}$$

The reason for these terms is based on the consideration of the similarity between the energy density and Lagrangian of the system. The energy density can be regarded as the Lagrangian parameterized with three parameter I_1, I_2 and J with each of them possibly constrained by $I_1 - 3 = 0$, $I_2 - 3 = 0$ and $J - 1 = 0$. Thus, according to the Analytical Mechanics, the Lagrangian of the constrained system can be written as the sum of the product of Lagrangian multiplier and the constraint. Hence the energy density is written as above.

2.2 Calculation of Cauchy Stress Tensor

The calculation is displayed in following

$$\begin{aligned}\boldsymbol{\sigma} &= \frac{2}{J}\boldsymbol{B}\cdot\frac{\partial W}{\partial \boldsymbol{B}}\\ &= \frac{1}{J}\boldsymbol{B}\cdot(C_1\frac{\partial\,\mathrm{tr}(\boldsymbol{B})}{\partial \boldsymbol{B}} + C_2\frac{\partial[(\mathrm{tr}\boldsymbol{B})^2 - \mathrm{tr}(\boldsymbol{B}^2)]}{\partial 2\boldsymbol{B}} + D\frac{\partial \det \boldsymbol{B}}{\partial \boldsymbol{B}})\\ &= \frac{1}{J}\boldsymbol{B}\cdot[C_1\boldsymbol{E} + C_2(I_1 - \boldsymbol{B}) + DJ\boldsymbol{B}^{-1}]\\ &= \frac{1}{J}[C_1\boldsymbol{B} - C_2(\boldsymbol{B}\cdot\boldsymbol{B} - I_1\boldsymbol{B} + I_2\boldsymbol{E}) + (D - \frac{I_2}{J})J\boldsymbol{E}]\\ &= \frac{1}{J}(C_1\boldsymbol{B} - C_2\boldsymbol{B}^{-1} + pJ\boldsymbol{E})\end{aligned} \tag{13}$$

Where the last equality is based on Cayley-Hamilton theorem[8]. And in the second step, the coefficient 2 is put into the material coefficient.

For incompressible material like rubber, there is $J = 1$, thus the result is

$$\boldsymbol{\sigma} = C_1\boldsymbol{B} - C_2\boldsymbol{B}^{-1} + p\boldsymbol{E} \tag{14}$$

3 Application to a Balloon

We now consider a balloon as a spherical incompressible hyperelastic rubber. Hence, the above analysis is suitable for the analysis of a balloon.[9]

First, we write the deformation gradient of a spherical balloon in the diagonal representation as

$$\boldsymbol{F} = \begin{bmatrix}\varepsilon & 0 & 0\\ 0 & \varepsilon & 0\\ 0 & 0 & \varepsilon^{-2}\end{bmatrix} \tag{15}$$

Where $\varepsilon = r/r_0$ is the stretch ratio. Thus, the left Cauchy-Green deformation tensor becomes

$$\boldsymbol{B} = \begin{bmatrix}\varepsilon^2 & 0 & 0\\ 0 & \varepsilon^2 & 0\\ 0 & 0 & \varepsilon^{-4}\end{bmatrix} \tag{16}$$

Hence, the Cauchy stress tensor will be

$$\boldsymbol{\sigma} = \begin{bmatrix}\sigma & 0 & 0\\ 0 & \sigma & 0\\ 0 & 0 & 0\end{bmatrix} = C_1\begin{bmatrix}\varepsilon^2 & 0 & 0\\ 0 & \varepsilon^2 & 0\\ 0 & 0 & \varepsilon^{-4}\end{bmatrix} + p\begin{bmatrix}1 & 0 & 0\\ 0 & 1 & 0\\ 0 & 0 & 1\end{bmatrix} - C_2\begin{bmatrix}\varepsilon^{-2} & 0 & 0\\ 0 & \varepsilon^{-2} & 0\\ 0 & 0 & \varepsilon^{4}\end{bmatrix} \tag{17}$$

Thus we will have

$$\sigma = C_1(\varepsilon^2 - \varepsilon^{-4}) + C_2(\varepsilon^4 - \varepsilon^{-2}) \tag{18}$$

And according to

$$\delta p \cdot \pi r^2 = \sigma \cdot 2\pi r d \tag{19}$$

With the help of $dr^2 = d_0 r_0^2$, we get the pressure difference of the inside and outside of the balloon

$$\delta p = 2C_1\frac{d_0}{r_0}(\varepsilon^{-1} - \varepsilon^{-7})(1 + \frac{C_2}{C_1}\varepsilon^2) \tag{20}$$

And according to the statistics we now have, i.e., $C_1 = 3$, $C_2 = 0.3$ and $d_0/r_0 = 0.008$, we plot the theoretical curve of pressure difference

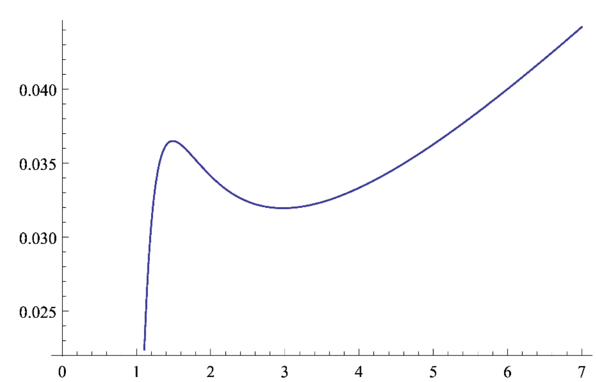

Figure 1: Theoretical Curve

If we consider the balloon as a sphere-like membrane instead of a strictly spherical one, the stretches of the two directions will not be equal. Instead, they will be proportional. Thus, we write the deformation gradient of as

$$\boldsymbol{F} = \begin{bmatrix} \varepsilon & 0 & 0 \\ 0 & \lambda\varepsilon & 0 \\ 0 & 0 & \lambda^{-1}\varepsilon^{-2} \end{bmatrix} \tag{21}$$

Where $\varepsilon = r/r_0$ is the stretch ratio. Thus, the left Cauchy-Green deformation tensor becomes

$$\boldsymbol{B} = \begin{bmatrix} \varepsilon^2 & 0 & 0 \\ 0 & \lambda^2\varepsilon^2 & 0 \\ 0 & 0 & \lambda^{-2}\varepsilon^{-4} \end{bmatrix} \tag{22}$$

Here, it can be manifestly seen that the stress of the two direction will not be equal, so we need to use different signs to denote them. Usually, the direction of the two directions will be set as the perpendicular to and along with the mouth of the balloon, and due to the nonlinearity of the stress-strain relation, they are likely not to be proportional. Hence, the Cauchy stress tensor will be

$$\boldsymbol{\sigma} = \begin{bmatrix} \sigma' & 0 & 0 \\ 0 & \sigma & 0 \\ 0 & 0 & 0 \end{bmatrix} = C_1 \begin{bmatrix} \varepsilon^2 & 0 & 0 \\ 0 & \lambda^2\varepsilon^2 & 0 \\ 0 & 0 & \lambda^{-2}\varepsilon^{-4} \end{bmatrix} + p \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} - C_2 \begin{bmatrix} \varepsilon^2 & 0 & 0 \\ 0 & \lambda^{-2}\varepsilon^{-2} & 0 \\ 0 & 0 & \lambda^2\varepsilon^4 \end{bmatrix} \tag{23}$$

Thus we will have

$$\sigma = C_1(\lambda^2\varepsilon^2 - \lambda^{-2}\varepsilon^{-4}) + C_2(\lambda^2\varepsilon^4 - \lambda^{-2}\varepsilon^{-2}) \tag{24}$$

And according to

$$\delta p \cdot \pi r^2 = \sigma \cdot 2\pi r d \tag{25}$$

With the help of $dr^2 = d_0 r_0^2$, we get the pressure difference of the inside and outside of the balloon

$$\delta p = 2C_1 \frac{d_0}{r_0}(\lambda^2\varepsilon^{-1} - \lambda^{-2}\varepsilon^{-7})(1 + \frac{C_2}{C_1}\varepsilon^2) \tag{26}$$

We need to notice that the radius here should be the radius of the largest cross-section circle which is perpendicular to the direction of the stress σ.

According to the statistics we now have, i.e., $C_1 = 3$, $C_2 = 0.3$, $r_0 = 83$mm and $d_0 = 0.1497$mm, we plot the theoretical curve of pressure difference

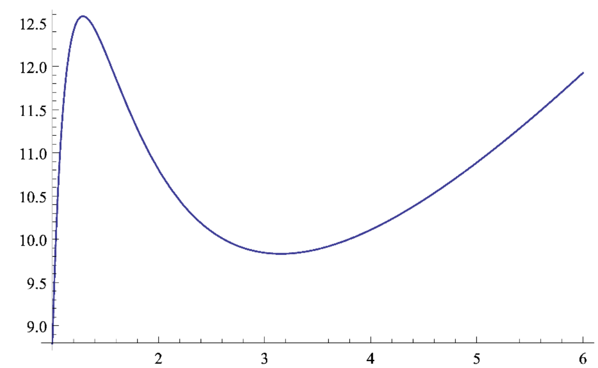

Figure 2: Theoretical Curve

References

[1] Wikipedia, *Finite Strain Theory*

[2] Wikipedia, *Young's Modulus*

[3] Wikipedia, *Strain Energy Density Function*

[4] Wikipedia, *Hyperelastic Material*

[5] Wikipedia, *Invariants of Tensors*

[6] Wikipedia, *Mooney-Rivlin Model*

[7] Wikipedia, *Polynomial (Hyperelastic Model)*

[8] Wikipedia, *Cayley-Hamilton theorem*

[9] Rubber and Rubber Balloons