Theoretical Analysis of Magnus Lift

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March 14, 2016

1 Classical Theory

The classical theory is constructed on some strict and ideal situations, and can lead to a basic analytical result although may not be quite practical. However, we still need the result for analysis for more complex situations will make the theoretical analysis unmeaningful.

The construction will be organized as follows: first, a velocity distribution of the air along a rotating cylinder will be calculated; second, an ideal steady flow across a static cylinder will be analyzed; Add this two parts together so that it can be formed a superposed air flow; Calculate the pressure difference caused by the flow and a lift can be gained.

1.1 Velocity Distribution

This section is to calculate the velocity distribution of the air outside a spinning cylinder. Firstly, we have the Navier-Stokes equation

$$\frac{\partial u^i}{\partial t} + u^j \nabla_j u^i - \mu \nabla^j \nabla_j u^i = -\frac{1}{\rho} \nabla^i p + g^i \tag{1}$$

And consider the steady condition which annihilates the dynamic part

$$u^j \nabla_j u^i - \mu \nabla^j \nabla_j u^i = -\frac{1}{\rho} \nabla^i p \tag{2}$$

And consider that the velocity has only the component of direction φ, i.e., $\vec{u} = (0, u, 0)$, as well as the boundary condition

$$\begin{aligned} \vec{u}(r = R) &= \vec{\omega} \\ \vec{u}(r \to \infty) &= 0 \end{aligned} \tag{3}$$

Where ω is the angular velocity of the closest air layer that is capable to superpose with the outer air stream, thus the solution will be

$$\vec{u} = \frac{\vec{\omega} R^2}{r^2} \tag{4}$$

Which is the result of this section.

1.2 Steady Flow outside Cylinder

This section is to calculate the steady flow passing a static cylinder. Here we need to use the stream line theory. First introduce the stream function vector as

$$\vec{\psi} = \vec{\nabla} \times \vec{v} \tag{5}$$

Where $\vec{\psi}$ is the stream function vector and $\vec{v}$ is the velocity.

According to reference [1], the stream function of the steady flow along the x-axis is

$$\psi = vy \tag{6}$$

And the stream function of a doublet is

$$\psi = -vy \frac{R^2}{r^2} \tag{7}$$

Thus, add this two parts up and the stream function of the steady flow passing the cylinder will be gained

$$\psi = vy \left(1 - \frac{R^2}{r^2}\right) \tag{8}$$

Hence the φ component of the velocity will be

$$u = \frac{1}{r} \frac{\partial \psi}{\partial r} = v \left(1 + \frac{R^2}{r^2}\right) \sin \varphi \tag{9}$$

Therefore the flow on the surface of the cylinder will be

$$u = \frac{2v \sin \varphi}{r} \tag{10}$$

Or written under the vector representation

$$\vec{u} = 2 \frac{\vec{r} \times \vec{v}}{r^2} \tag{11}$$

Which is the final conclusion of this section

1.3 Classical Lift

This section is to calculate the theoretical classical lift of the spinning and moving cylinder. First there is the superposed velocity

$$\vec{\eta} = \left(\frac{\vec{\omega} R^2}{r^2} + 2\frac{\vec{r} \times \vec{v}}{r^2} \right) \times \vec{r} \tag{12}$$

Thus the dynamic pressure will be

$$\begin{aligned} p &= \frac{1}{2}\rho\vec{\eta}\cdot\vec{\eta} \\ &= \frac{1}{2}\rho\vec{\omega}^2\frac{R^4}{r^2} + 2\rho\frac{R^2}{r}\vec{\omega}\cdot(\vec{e_r}\times\vec{v}) + 2\rho(\vec{e_r}\times\vec{v})^2 \end{aligned} \tag{13}$$

Calculation will be as follows

$$\begin{aligned} F_\mu =& \iint \delta p e^r_\mu \, dA_r \\ =& \iint \delta p e^r_\nu \varepsilon^\nu_a \varepsilon^a_\mu \, dA_r \\ =& 2\iint \rho R(\vec{v}\times\vec{\omega})_z \sin^2\varphi \varepsilon^z_\mu R d\varphi dy \\ &+ 2\iint \rho R(\vec{v}\times\vec{\omega})_z \sin\varphi\cos\varphi \varepsilon^z_\mu R d\varphi dy \\ &+ \frac{1}{2}\iint \rho R^2\vec{\omega}^2 \sin\varphi R d\varphi dy \\ =& 2\pi\rho R^2 h(\vec{v}\times\vec{\omega})_z\varepsilon^z_\mu \\ =& 2\pi\rho R^2 h(\vec{v}\times\vec{\omega})_\mu \end{aligned} \tag{16}$$

Hence the conclusion is

$$\vec{F} = 2\pi\rho R^2 h \cdot \vec{v}\times\vec{\omega} \tag{17}$$

2 Practical Theory

The above classical theory is a mathematically rigorous one, and is also a simple and basic theory of Magnus effect. However, it is based on some assumptions such as that the boundary layer is always laminar and that there will not be any air separation, which is almost impossible in a practical situation. Thus, there must be some modifications on this classical theory.

Usually, the lift of any kind of airfoil can be written as the product of the dynamic pressure, the effective airfoil area and a constant, i.e.

$$F = \frac{1}{2}\rho v^2 SC \tag{18}$$

Thus, define the lift coefficient as

$$C_L = \frac{F}{\frac{1}{2}\rho v^2 S} \tag{19}$$

Hence the pressure difference

$$\begin{aligned} \delta p &= p(\infty) - p(R) \\ &= -2pR(\vec{v}\times\vec{\omega})^z\vec{\varepsilon}_z\cdot\vec{e}_r - \frac{1}{2}\vec{\omega}^2R^2 \\ &= -2pR(\vec{v}\times\vec{\omega})^z\sin\varphi - \frac{1}{2}\vec{\omega}^2R^2 \end{aligned} \tag{14}$$

Thus the theoretical Magnus force

$$F_\mu = -\iint \delta p \, dA_\mu \tag{15}$$

In the case of the cylinder, there will be $F = 2\pi\rho R^2 hv\omega$ and $S = 2ab$, and hence, theoretically

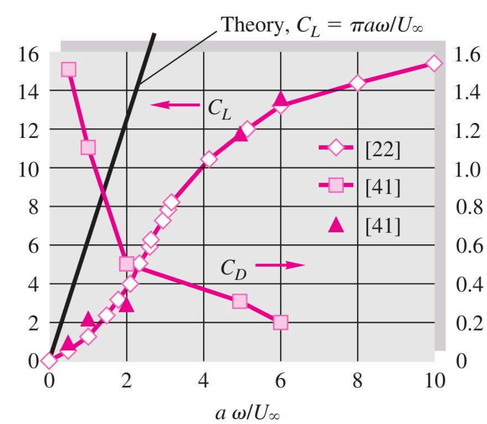
$$C_L = 2\pi x \tag{20}$$

Where

$$x = \frac{\omega R}{v} \tag{21}$$

is the dimensionless variable.

Nevertheless, experimentally



Therefore, the influence of the turbulence, air separation and wingtip vortex can be much larger than expected. Thus, it is very important to have a correction factor on the lift coefficient, i.e.

$$C_L = 2\pi x\eta \tag{22}$$

Where the correction factor η can represent the influence led by turbulence, air separation, wingtip loss and other kind of practical factors.

References

[1] Foundations of Aerodynamics

[2] Wikipedia, *Navier-Stokes Equation*

[3] Fluid Mechanics

[4] An Introduction to Fuid Dynamics

[5] Wikipedia, *Bernoulli's Principle*

[6] Wikipedia, *Euler Equation*

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[8] Wikipedia, *Kutta-Joukowski Theorem*

[9] Wikipedia, *Magnus Effect*

[10] Wikipedia, *Stream Function*

[11] Wikipedia, *Vortex*