Theoretical Analysis of Halo

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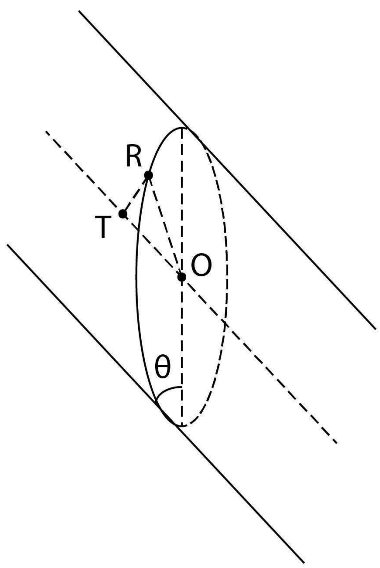
Set up a coordinate system at the centre of the cross section of the wire, as is shown in the graph



We have the curve of light beam:

$$z = t \tag{1}$$

The centre line

$$l : \begin{cases} y = s\cot\theta \\ z = s \end{cases} \tag{2}$$

And the oval

$$o : \begin{cases} x = r\cos t \\ y = r\csc\theta\sin t \end{cases} \tag{3}$$

Thus, vector $\vec{TR}$ can be written as

$$\vec{TR} = (r\cos t, r\csc\theta\sin t - s\cot\theta, -s) \tag{4}$$

According to

$$\vec{l} \cdot \vec{TR} = 0 \tag{5}$$

We'll have

$$s = r\sin t\cos\theta \tag{6}$$

Thus

$$\vec{n} = \vec{TR} = (r\cos t, r\sin t\sin\theta, -r\sin t\cos\theta) \tag{7}$$

Suppose the direction tangent vector of the incident line is $\vec{i}$ and the direction tangent vector of the reflection line is $\vec{r}$, then we will have the following equations

$$\begin{aligned} \cos\left\langle \vec{i}, \vec{n} \right\rangle &= \cos\langle \vec{r}, \vec{n}\rangle \\ \vec{r} \cdot (\vec{i} \times \vec{n}) &= 0 \end{aligned} \tag{8}$$

Hence we get

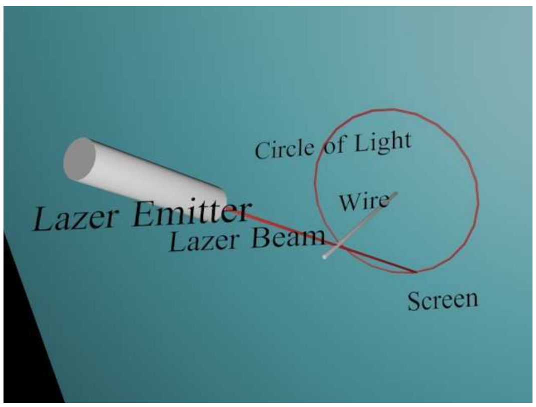
$$r : \begin{cases} x \cdot n_1 + y \cdot n_2 + (z - |l|) \cdot n_3 = 0 \\ y \cdot n_1 - x \cdot n_2 = 0 \end{cases} \tag{9}$$

Where x, y and z are components of the reflection light vector, and $l = \sqrt{x^2 + y^2 + z^2}$ is the length of the reflection light vector. It can be clearly seen that the solution of equation (9) produces a cone.

If the screen is perpendicular to the wire, the formulation of the screen plane will be

$$y\cos\theta + z\sin\theta - d = 0 \tag{10}$$

Where d is the distance of the screen to the incidence point.



Thus the cross-section of the cone (9) by the screen can be formulated as

$$\begin{cases} x = 2d\sin t\cos t\cot\theta \\ y = 2d\sin^2 t\cos\theta \\ z = d(\cos^2 t\csc\theta - \sin^2 t\cos\theta\cot\theta + \sin^2 t\sin^2\theta) \end{cases} \tag{11}$$

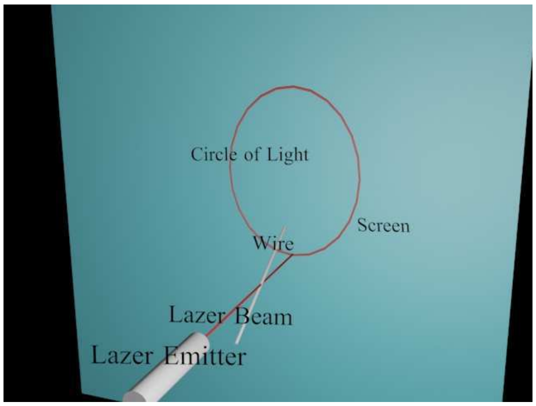
Eliminate the parameter and we get

$$x^2 + (y - d\cos\theta)^2 + (z - d\sin\theta)^2 = d^2\cot^2\theta \tag{12}$$

Therefore, it can be clearly seen that the cross-section at the screen which is perpendicular to the wire is a circle with center at $(0, d\cos\theta, d\sin\theta)$ and radius being $d\cot\theta$.

To generate a simpler version, suppose the screen is formulated by

$$z = z_0 \tag{13}$$



Then we will get

$$\begin{cases} x = \dfrac{2\cos t\sin\theta}{\cos^2 t - \cos 2\theta\sin^2 t} z_0 \\ y = \dfrac{2\sin\theta\cos\theta\sin^2 t}{\cos^2 t - \cos 2\theta\sin^2 t} z_0 \end{cases} \tag{14}$$

Eliminate the parameter and we get

$$x^2\sin^2\theta - y^2\cos(2\theta) - y \cdot 2z_0\cos\theta\sin\theta = 0 \tag{15}$$

Which is written in the form of

$$Ax^2 + Bxy + Cy^2 + Dx + Ey + F = 0 \tag{16}$$

Hence it is manifest that the cross-section is a conic curve.

According to

$$e = \sqrt{\frac{2\sqrt{(A-C)^2 + B^2}}{\eta(A+C) + \sqrt{(A-C)^2 + B^2}}} \tag{17}$$

Where

$$\eta = -\text{sign}\{\begin{bmatrix} A & B/2 & D/2 \\ B/2 & C & E/2 \\ D/2 & E/2 & F \end{bmatrix}\} \tag{18}$$

Thus, calculation shows

$$e = \cot\theta \tag{19}$$

Therefore it is found that the eccentricity of the cross-section conic curve is related and only related to the dip angle of the wire.

Next, it is going to be calculated the light intensity of the circle. The major method is to project the reflection cross-section, i.e., the plane perpendicular to the laser beam which has a uniform light intensity, to the cross-section of the screen plane.

Hence, suppose the reflection cross-section border is formulated as

$$\begin{cases} x = r\cos t \\ y = r\sin t\csc\theta \end{cases} \tag{20}$$

And the screen cross-section border is formulated as

$$\begin{cases} x = r\cos t \\ y = r\sin t\csc\varphi \end{cases} \tag{21}$$

Therefore, there will be

$$dl = r\sqrt{\cos^2 t\csc^2\theta + \sin^2 t}\, dt \tag{22}$$

And

$$dl' = r\sqrt{\cos^2 t\csc^2\varphi + \sin^2 t}\, dt \tag{23}$$

According to

$$\sigma dl = \sigma' dl' \tag{24}$$

Where σ and σ' represents the light intensity density at reflection cross-section and screen cross-section respectively, there will be

$$\sigma' = \sqrt{\frac{\csc^2\theta + \tan^2 t}{\csc^2\varphi + \tan^2 t}}\sigma \tag{25}$$

Which is the conclusion of light intensity analysis.

References

[1] Wikipedia, *Snell's Law*

[2] Wikipedia, *Reflection*

[3] Wikipedia, *Cross Section (Physics)*

[4] Wikipedia, *Luminous Intensity*

[5] Wikipedia, *Laser*