Notes on U(1) Gauge Field

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Abstract This is a brief notes on construction of $\mathrm{U}(1)$ gauge field.

Conventions and Notation

Unit system Natural unit system with

$$c = \hbar = 1, \quad \mu_0 = \varepsilon_0 = 1$$

Einstein Summation Convention Repeating covariant and contravariant indices refers to summation.

Background Construction

To construct a U(1) gauge field theory, we need

> **Good enough base manifold** M
> **Matrix Lie Group** $\mathrm{U}(1)$
> **Principal Bundle** $P = M \times \mathrm{U}(1)$

In ordinary Quantum Field Theory in flat Minkowski spacetime, it is required to that there is a Minkowski metric $\eta_{\mu\nu}$ defined on base manifold M.

Field Construction

Usually, the gauge field is constructed as the connection of principal bundle. There are in total three equivalent definition of connection of principal bundle.

> **Definition.** The **connection** of principal bundle $P(M, G)$ is a C^∞ n-dimensional distribution H_p such that
> (a) $\mathrm{T}_p P = \mathrm{V}_p \oplus \mathrm{H}_p$ (b) $R_{g*}[H_p] = H_{pg}$

> **Definition.** The **connection** of principal bundle $P(M, G)$ is a C^∞ $\mathfrak{g}$-valued 1-form ω such that
> (a) $\omega_p(A_p^*) = A$ (b) $\omega_{pg}(R_{g*}X) = \mathrm{ad}_{g^{-1}}\omega_p(X)$

> **Definition.** The **connection** of principal bundle $P(M, G)$ with G being a matrix group is a C^∞ $\mathfrak{g}$-valued 1-form ω_U for each local trivialization T_U such that if g is the transition

function between local trivialization T_U and T_V there is

$$\omega_V = g^{-1}\omega_U g + g^{-1}\mathrm{d}g$$

Usually, the gauge field strength is defined as the curvature of the connection.

Definition The **curvature** of connection ω is defined as

$$\Omega = \mathrm{d}\omega + \frac{1}{2}[\omega, \omega]$$

For U(1) gauge field theory, we identify the relationship between the connection ω (in the third definition) and the gauge potential A as

$$\omega_\mu = eA_\mu$$

where e is the coupling constant, as well as the relationship between the relationship between the curvature Ω and the gauge field strength F as

$$\Omega_{\mu\nu} = eF_{\mu\nu}$$

Thus, we have completed the construction of gauge field.

Equations of Motion

Construct the Lagrangian of U(1) gauge field as

$$\mathcal{L} = -\frac{1}{4}F_{\mu\nu}F^{\mu\nu}$$

As usual, action type-I (see Structure of Physics) is written as

$$\mathcal{I} = \int \mathcal{L}\varepsilon$$

where ε is the adapted volume element. According to

$$\iota\delta\mathcal{I} = 0$$

(see Structure of Physics) This directly leads to

$$\partial^\mu\partial_\mu A_\nu - \partial^\mu\partial_\nu A_\mu = 0$$

This is not an obvious wave equation. However, if we take the Lorentz gauge which means

$$\partial_\mu A^\mu = 0$$

Then, under the circumstance of flat spacetime, there will be

$$\partial^\mu\partial_\mu A_\nu = 0$$

which is an obvious wave equation.