Notes on Theoretical Mechanics

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Conventions and Notation

Unit System Gaussian natural unit system with

$$c = \hbar = 1 \quad \text{and} \quad \varepsilon_0^{-1} = \mu_0 = 4\pi$$

Index System Einstein summation convention with

$$a_i b^i = \sum_i a_i b^i$$

Spacetime Background

Base manifold $(\mathbb{R}^4, \delta_{\mu\nu})$ with spacetime translation symmetry.
All discussion is based on Newtonian classical mechanics.

Particle Kinematics

Basic Concepts

Basic Concept	Definition
Particle	A smooth world line C pointed to the future.
Reference frame	A set of observers satisfying that for any point in spacetime there exists unique observer passing through.
Degrees of freedom	Dimension of the system.
Kinematic equation	Particle parametrized by absolute time.
Trajectory	The image of the particle.
Displacement	$\Delta\vec{r} = \vec{r}(t + \Delta t) - \vec{r}(t)$

Path	Length of the particle between (t, t')
Velocity	Tangent vector of the particle
Acceleration	Directional derivative along the particle $a^\mu = u^\nu \partial_\nu u^\mu$

Coordinate System

Coordinate System	Metric
Rectangular Coordinate System	$\mathrm{d}s^2 = \mathrm{d}x^2 + \mathrm{d}y^2 + \mathrm{d}z^2$
Polar Coordinate System	$\mathrm{d}s^2 = \mathrm{d}r^2 + r^2 \mathrm{d}\theta^2$
Spherical Coordinate System	$\mathrm{d}s^2 = \mathrm{d}r^2 + r^2 \mathrm{d}\theta^2 + r^2 \sin^2\theta \mathrm{d}\varphi^2$
Cylindrical Coordinate System	$\mathrm{d}s^2 = \mathrm{d}r^2 + r^2 \mathrm{d}\theta^2 + \mathrm{d}z^2$

Christopher Symbol of Coordinate System

(r, θ) Series

$$\Gamma^r{}_{\theta\theta} = -r, \quad \Gamma^\theta{}_{r\theta} = r^{-1}$$

(r, θ, φ) Series

$$\begin{gathered}
\Gamma^r{}_{\theta\theta} = -r, \quad \Gamma^r{}_{\varphi\varphi} = -r\sin^2\theta \\
\Gamma^\theta{}_{r\theta} = r^{-1}, \quad \Gamma^\theta{}_{\varphi\varphi} = -\sin\theta\cos\theta \\
\Gamma^\varphi{}_{r\varphi} = r^{-1}. \quad \Gamma^\varphi{}_{\theta\varphi} = \cot\theta
\end{gathered}$$

Velocity and Acceleration in Coordinate System

In the following context, components are coordinate components and $\{e_\mu\}$ refers to orthonormal basis.

Rectangular Coordinate System

Quantity	Expression
Velocity	$v^x = \dot{x}, \quad v^y = \dot{y}, \quad v^z = \dot{z}$
Acceleration	$a^x = \ddot{x}, \quad a^y = \ddot{y}, \quad a^z = \ddot{z}$

Thus

$$\vec{v} = \dot{x}\vec{e}_x + \dot{y}\vec{e}_y + \dot{z}\vec{e}_z$$
$$\vec{a} = \ddot{x}\vec{e}_x + \ddot{y}\vec{e}_y + \ddot{z}\vec{e}_z$$

Polar Coordinate System

Quantity	**Expression**
Velocity	$v^r = \dot{r}, \quad v^\theta = \dot{\theta}$
Acceleration	$a^r = \ddot{r} - r\dot{\theta}^2, \quad a^\theta = \ddot{\theta} + 2\dot{r}\dot{\theta}/r$

Thus

$$\vec{v} = \dot{r}\vec{e}_r + r\dot{\theta}\vec{e}_\theta$$
$$\vec{a} = (\ddot{r} - r\dot{\theta}^2)\vec{e}_r + (r\ddot{\theta} + 2\dot{r}\dot{\theta})\vec{e}_\theta$$

Spherical Coordinate System

Quantity	**Expression**
Velocity	$v^r = \dot{r}, \quad v^\theta = \dot{\theta}, \quad v^\varphi = \dot{\varphi}$
Acceleration	$a^r = \ddot{r} - r\dot{\theta}^2 - r\dot{\varphi}^2 \sin^2\theta$ $a^\theta = \ddot{\theta} + 2\dot{r}\dot{\theta}/r - \dot{\varphi}^2 \sin\theta\cos\theta$ $a^\varphi = \ddot{\varphi} + 2\dot{r}\dot{\varphi}/r + 2\dot{\theta}\dot{\varphi}\cot\theta$

Thus

$$\vec{v} = \dot{r}\vec{e}_r + r\dot{\theta}\vec{e}_\theta + r\dot{\varphi}\sin\theta\vec{e}_\varphi$$
$$\vec{a} = (\ddot{r} - r\dot{\theta}^2 - r\dot{\varphi}^2\sin^2\theta)\vec{e}_r + (r\ddot{\theta} + 2\dot{r}\dot{\theta} - \dot{\varphi}^2 r\sin\theta\cos\theta)\vec{e}_\theta$$
$$+ (r\ddot{\varphi}\sin\theta + 2\dot{r}\dot{\varphi}\sin\theta + 2r\dot{\theta}\dot{\varphi}\cos\theta)\vec{e}_\varphi$$

Natural Coordinates (Proper Coordinates)

Velocity: $\vec{v} = \dot{s}\vec{e}_t$
Acceleration:

$$\vec{a} = \ddot{s}\vec{e}_t + \frac{\dot{s}}{\rho}\vec{e}_n$$

where s is the path of the particle

Rigid Body Kinematics

Rigid Body

Rigid body is a smooth world line C pointed to the future with a 3-frame.

Degrees of Freedom

The **degrees of freedom of a free rigid body** is $s = 6$.

Reference Point

The **reference point** intersection point of a rigid body world line and a simultaneous plane.
Any point of the rigid body can be expressed by the reference point and a space vector at reference point.

Basic Motion

Translation If any space vector at the reference point is static, then the rigid body is under a translation.
Rotation If there exists a space vector at the reference point that is static, then the rigid body is under a rotation.

> **Theorem** Any motion of the rigid body can be decomposed into translation and rotation.

Angular Velocity

Rigid body is rotating if there exists a 2-form $\Omega_{\mu\nu}$ such that for any space vector w^μ at reference point, there is

$$u^\nu \partial_\nu w^\mu = \Omega^\mu{}_\nu w^\nu$$

The 2-form $\Omega_{\mu\nu}$ is the **angular velocity form** of the rigid body.
The dual of $\Omega_{\mu\nu}$ denoted by ω_μ is the **angular velocity** of the rigid body.

Techniques of Special Motion

Fix-axis Rotation The angular velocity is static in direction.
Velocity of any point $\vec{v}_p = \vec{\omega} \times \vec{r}_p$
Acceleration of any point $\vec{a}_p = \dot{\vec{\omega}} \times \vec{r}_p + \vec{\omega} \times (\vec{\omega} \times \vec{r}_p)$

Plane Parallel Motion Translation is restricted in a plane and the direction of angular velocity is restricted to be perpendicular to the plane.
Velocity of any point $\vec{v}_p = \vec{v}_{rp} + \vec{\omega} \times \vec{r}_p$
Acceleration of any point $\vec{a}_p = \vec{a}_{rp} + \dot{\vec{\omega}} \times \vec{r}_p + \vec{\omega} \times (\vec{\omega} \times \vec{r}_p)$

Typical Model Rotation without slipping

Fixed-point Rotation There exists a point that is static.
Any instant of fixed-point rotation can be regarded as a fixed-axis rotation.

Particle Dynamics

Dynamical Equation

The dynamical equation of Newtonian classical mechanics is

$$m\ddot{\vec{r}} = \sum_{i=1}^{n} \vec{F}_i(\vec{r}, \dot{\vec{r}}, t)$$

Dynamics with Constraint

Adding constraint force and constraint equations.

Momentum Theorem

> **Momentum Theorem**
>
> $$\sum_{i=1}^{n} F_i^{\mu} = u^{\nu} \partial_{\nu} p^{\mu}$$
>
> **Conservation**
>
> $$\sum_{i=1}^{n} F_i^{\mu} = 0 \Rightarrow p^{\mu} = \text{constant}$$

Angular Momentum and Torque

Position Displacement $\vec{r}$ The space vector corresponding to the position of the particle and another point.
Torque $\vec{M} = \vec{r} \times \vec{F}$
Angular Momentum $\vec{L} = \vec{r} \times \vec{p}$

> **Angular Momentum Theorem**
>
> $$\sum_{i=1}^{n} M_i^{\mu} = u^{\nu} \partial_{\nu} L^{\mu}$$
>
> **Conservation**

$$\sum_{i=1}^{n} M_i^\mu = 0 \Rightarrow L^\mu = \text{constant}$$

Theorem of Kinetic Energy and Energy Conservation Law

Theorem of Kinetic Energy

$$\mathrm{d}T = \delta W$$

Energy Conservation Law

$$\partial^\mu T_{\mu\nu} = 0$$

Conservative Force

If $\vec{\nabla} \times \vec{F} = 0$, then $\vec{F} = \vec{\nabla} U$. In this circumstance, force $\vec{F}$ is conservative and U is the potential of the force.

Potential Curve and Stability

Omitted.

Particle Dynamics under Any Observer

Particle Kinematics under Any Observer

Set Z^μ the 4-velocity of any observer, and S' its proper coordinate system.

Velocity

$$u^\mu = u'^\mu + Z^\mu + \Omega^\mu{}_\nu r^\nu$$

Acceleration

$$a^\mu = a'^\mu + A^\mu + u^\rho \partial_\rho \Omega^\mu{}_\nu r^\nu + \Omega^\mu{}_\rho \Omega^\rho{}_\nu r^\nu + 2\Omega^\mu{}_\nu u'^\nu$$

within, $A^\mu = Z^\nu \partial_\nu Z^\mu$ is the 4-acceleration of the observer.

The **centrifugal acceleration** is defined as $\vec{a}_g = \vec{\omega} \times (\vec{\omega} \times \vec{r})$
The **Coriolis's acceleration** is defined as $\vec{a}_c = 2\vec{\omega} \times \vec{v}$

Dynamics

Centrifugal Force $\vec{F}_g = m\vec{\omega} \times (\vec{\omega} \times \vec{r})$
Centrifugal force is conservative.

Coriolis's Force $\vec{F}_c = 2m\vec{\omega} \times \vec{v}$
Coriolis's force does not have work.

Dynamics of Particle System

Internal Force and External Force

Force between particles are **internal force**. Internal force has the following properties

$$\sum_{j=1}^{n} F_j^{(int)} = 0, \qquad \sum_{j=1}^{n} M_j^{(int)} = 0$$

Momentum Theorem

Total Momentum $\vec{p} = \sum \vec{p}_i$

Momentum Theorem

$$\sum_{i=1}^{n} F^{\mu}_{i\,(ext)} = u^{\nu}\partial_{\nu}p^{\mu}$$

Conservation

$$\sum_{i=1}^{n} F^{\mu}_{i\,(ext)} = 0 \Rightarrow p^{\mu} = \text{constant}$$

Center of Mass

Center of Mass is defined as

$$x_c^{\mu} = \frac{1}{m}\sum_{i=1}^{n} m_i x_i^{\mu}$$

Dynamical Equation of Center of Mass

$$\sum_{i=1}^{n} \vec{F}_i^{(ext)} = m\ddot{\vec{x}}_c$$

Angular Momentum Theorem

Total Angular Momentum $\vec{L} = \sum \vec{L}_i$

Momentum Theorem

$$\sum_{i=1}^{n} M^{\mu}_{i\,(ext)} = u^{\nu}\partial_{\nu}L^{\mu}$$

Conservation

$$\sum_{i=1}^{n} M^{\mu}_{i\,(ext)} = 0 \Rightarrow L^{\mu} = \text{constant}$$

Kinetic Energy

Total Kinetic Energy

$$T = \sum_{i=1}^{n} T_i = \sum_{i=1}^{n} \frac{1}{2} m_i v_i^2$$

Koenig's Theorem

$$T = \frac{1}{2} m v_c^2 + \sum_{i=1}^{n} \frac{1}{2} m_i v_i'^2$$

Theorem of Kinetic Energy

Theorem of Kinetic Energy $\mathrm{d}T = \delta W^{(ext)} + \delta W^{(int)}$

Rotational Dynamics of Rigid Body

Rotational Kinetic Energy

$$T_r = \frac{1}{2} I \ddot{\varphi}$$

Analytical Mechanics

Action Variation Principle

Action type-S (see Structure of Physics)

$$S = \int L \mathrm{d}t$$

where L is the Lagrangian defined by

$$L = T - V$$

where T is the kinematic energy and V is the potential energy.

Principle of Virtual Work

At static equilibrium, there is

$$\delta V = Q_i \delta q^i = 0$$

Classification of Constraint

Complete: $f(q_i, t) = 0$
Steady: $f(q_i, \dot{q}_i) = 0$
Bilateral: $f = 0$
Unilateral: $f \leqslant 0$ or $f < 0$
Ideal: $Q_i \delta q^i = 0$ where Q_i are constraint forces.

Euler Equation

> **Euler Equation**
>
> $$\frac{\partial L}{\partial q^i} - \frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial L}{\partial \dot{q}^i}\right) = 0$$

Momentum and Energy

Momentum: $p_i = \partial L / \partial \dot{q}^i$
Energy: $H = p_i q^i - L$

Hamiltonian

The Hamiltonian is defined as

$$H = p_i q^i - L$$

Classical Poisson Bracket

> Classical Poisson Bracket is defined as
>
> $$\{f, g\} = \frac{\partial f}{\partial q^i}\frac{\partial g}{\partial p_i} - \frac{\partial f}{\partial p_i}\frac{\partial g}{\partial q^i}$$
>
> There is
>
> $$\frac{\mathrm{d}f}{\mathrm{d}t} = \frac{\partial f}{\partial t} + \{f, H\}$$

Canonical Equation

Canonical Equation

$$
\begin{aligned}
\dot{q}^i &= \{q^i, H\} = \frac{\partial H}{\partial p_i} \\
\dot{p}_i &= \{p_i, H\} = -\frac{\partial H}{\partial q^i}
\end{aligned}
$$

Central Force

Beanie Equation:

$$
-mh^2u^2\left(\frac{\mathrm{d}^2 u}{\mathrm{d}\theta^2} + u\right) = F
$$

where $u \equiv 1/r$ and $h \equiv r^2\dot{\theta}$.