Notes on First Quantization of Electromagnetic Field

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Abstract This is a brief notes on the first quantization of electromagnetic field, or U(1) gauge field.

Conventions and Notation

Unit system Natural unit system with

$$c = \hbar = 1, \quad \mu_0 = \varepsilon_0 = 1$$

Einstein Summation Convention Repeating covariant and contravariant indices refers to summation.

Motivation

In 1924, De Broglie proposed in his PhD thesis that all matter has a wave property with wavelength and momentum satisfying a relationship

$$\lambda = \frac{h}{p}$$

For a long time, this serves as a hypothesis of quantum theory. However, in modern Physics, we always hope that it is possible to reduce the number of independent hypotheses and gain more essential principles. Thus, in this essay, we are going to show that certain construction of the Lagrangian can automatically leads to the above relationship, and hence makes the De Broglie wave assumption a result of Lagrange mechanics. This essay mainly focuses on U(1) gauge field.

First Quantization Condition

We directly write down the first quantization condition

> **First Quantizaition Condition.** The first quantization condition is to construct the Lagrangian of U(1) gauge field as
>
> $$\mathcal{L} = -\frac{1}{4} F_{\mu\nu} F^{\mu\nu}$$

Energy-momentum Tensor

It is required that there will be spacetime translation symmetry for all high energy theories. Thus, according to Noether's theorem, there will be a corresponding conserved canonical energy-momentum tensor

$$T^C_{\mu\nu} = -\frac{\partial \mathcal{L}}{\partial \partial^\mu A_\lambda}\partial_\nu A_\lambda + \eta_{\mu\nu}\mathcal{L}$$

Therefore, according to the first quantization condition, the energy-momentum tensor of the electromagnetic field is

$$T^C_{\mu\nu} = F_{\mu\lambda}\partial_\nu A^\lambda - \frac{1}{4}\eta_{\mu\nu}F_{\rho\sigma}F^{\rho\sigma}$$

However, we find that this energy momentum tensor is not symmetric, which brings difficulties in defining the 4-momentum of electromagnetic field. Thus, we introduce the Belinfante-Rosenfield energy-momentum tensor

$$T^{\mu\nu} = T^{\mu\nu}_C - \frac{i}{2}\partial_\kappa\left[\frac{\partial \mathcal{L}}{\partial(\partial_\kappa A^\ell)}(\mathcal{J}^{\mu\nu})^\ell{}_m A^m - \frac{\partial \mathcal{L}}{\partial(\partial_\mu A^\ell)}(\mathcal{J}^{\kappa\nu})^\ell{}_m A^m - \frac{\partial \mathcal{L}}{\partial(\partial_\nu A^\ell)}(\mathcal{J}^{\kappa\mu})^\ell{}_m A^m\right]$$

This provides

$$T_{\mu\nu} = F_{\mu\lambda}F_\nu{}^\lambda - \frac{1}{4}\eta_{\mu\nu}F_{\rho\sigma}F^{\rho\sigma}$$

Plane Wave Solution

The equations of motion of U(1) gauge field under Lorentz gauge is written as

$$\partial^\mu\partial_\mu A_\nu = 0$$

Thus, the plane wave solution of this equation can be

$$A_\nu = NC_\nu \exp(-\mathrm{i}k^\mu x_\mu)$$

where C_ν is a dimensionless transverse ($C_\mu k^\mu = 0$) constant ($\partial_\mu C_\nu = 0$) normalized $C_\mu C^\mu = 1$ vector and N is a normalization factor. Recall that plane wave solution has a continuous spectrum which makes it unable to normalize. Hence, it is not a physical solution of an actual electromagnetic field. However, it is possible to perform a box normalization trick — select a global 3+1 decomposition and restrict the plane wave inside a box with volume V. Therefore, N should have a factor $V^{-1/2}$.

This is not enough. Let's do a dimension analysis to find the dimension of A_μ. The dimension of the Lagrangian is $[L]^{-4}$, and hence the dimension of $F_{\mu\nu}$ is $[L]^{-2}$, which results in the dimension of A_μ being $[L]^{-1}$. Consequently, a single $V^{-1/2}$ is not enough.

To resolve this, recall that electromagnetic field has a global U(1) symmetry. The corresponding Noether current can be written as

$$j_\mu = F_{\mu\lambda}A^\lambda = (\partial_\mu A_\lambda - \partial_\lambda A_\mu)A^\lambda$$

Remember that j^0 is the charge density and is required to be normalized. Investigating the above quantity, we find that the second term vanishes due to the transverse of electromagnetic field, and the first term gives a k^0. Thus, there should be a $(k^0)^{-1/2}$ in the normalization factor.

In conclusion, the plane wave solution should be written as

$$A_\nu = (k^0 V)^{-\frac{1}{2}} C_\nu \exp(-\mathrm{i}k^\mu x_\mu)$$

Equivalence with De Broglie's Assumption

Now, we use the spacetime translation Noether's current to calculate the total 4-momentum of electromagnetic plane wave. Define the 4-momentum as

$$P_\mu = \int \varepsilon T_{\mu 0}$$

Notice that the second term in $T_{\mu\nu}$ vanishes under plane wave solution. Hence

$$P_\mu = \int \varepsilon(\partial_\mu A_\lambda - \partial_\lambda A_\mu)(\partial_0 A^\lambda - \partial^\lambda A_0)$$

$$= (k^0 V)^{-1} \int \varepsilon(k_\mu C_\lambda - k_\lambda C_\mu)(k_0 C^\lambda - k^\lambda C_0)$$

Recall that $k_\mu k^\mu = 0$ and $k_\mu C^\mu = 0$, there is

$$P_\mu = \int \varepsilon V^{-1} k_\mu = k_\mu$$

Thus, we prove that the 4-momentum of a plane electromagnetic wave has a relationship with the wave number (and hence the wavelength). It is easy to verify that the above relation is just another form of De Broglie's assumption.

Since all solution of the equation of motion can be constructed through the plane wave solution, it can be easily verify that this formulation of Lagrangian reconstructs the first quantization process.

Conclusion

In this essay, we prove that De Broglie's first quantization condition in fact corresponds to a typical formulation of Lagrangian, and consequently, the first quantization no longer serves as an assumption but now a conclusion in Quantum Field Theory.